A Khovanov-theoretic invariant of bridge trisections

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Northeastern University
AMS Section Meeting
Khovanov homology is a TQFT:

- Links $\rightarrow$ vector spaces/modules
- Link cobordisms $\rightarrow$ linear maps

$\text{Kh}(U) \rightarrow \text{Kh}(U)$ is an isotopy invariant.
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$\text{Kh}(U) \rightarrow \text{Kh}(U)$ is an isotopy invariant.

**Theorem (Rasmussen; Tanaka)**

*This map can only tell you if $\Sigma$ has genus one or not.*
<table>
<thead>
<tr>
<th>Definition (Meier, Zupan)</th>
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<tbody>
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<td>A ((b, c))-bridge trisection diagram is a trio of trivial tangles ((t_1, t_2, t_3)) on (2b) points so that (t_i \bar{t}_j) is a (c)-component unlink.</td>
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<td><em>Bridge trisection diagrams represent bridge-trisected surfaces in (S^4).</em></td>
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<td><em>Two diagrams represent the same isotopy class of surface if and only if they are related by a sequence of special moves.</em></td>
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From chain complexes to algebras.

Cobordism $t_1 \tilde{t}_2 \coprod t_2 \tilde{t}_3 \rightarrow t_1 \tilde{t}_3$ induces a map

$$m : \text{CKh}(t_1 \tilde{t}_2) \otimes \text{CKh}(t_2 \tilde{t}_3) \rightarrow \text{CKh}(t_1 \tilde{t}_3).$$
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In general, $m : \text{CKh}(t_i \bar{t}_j) \otimes \text{CKh}(t_j \bar{t}_k) \rightarrow \text{CKh}(t_i \bar{t}_k)$. 
The invariant: round 1

\[ t = (t_1, t_2, t_3) \] a bridge trisection of \( \Sigma \).

Define \( A(t) = \bigoplus_{i,j} \text{CKh}(t_i \bar{t}_j) \).

**Proposition (Khovanov?)**

\((A(t), \partial_{\text{CKh}}, m)\) is an (associative) differential graded algebra. Its chain homotopy type is an invariant of the trisection of \( \Sigma \).
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**Proposition (Khovanov?)**

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**Proposition (S.)**

For connected \( \Sigma \), the algebra \((H(A(t)), m_\ast)\) is determined by \( b \) and \( c \), and therefore by \( g(\Sigma) \).
Link $L$ with diagram $\mathcal{D}$

Filtered chain complex $(\text{CSz}(\mathcal{D}), \partial)$ where

- $\text{CSz}(\mathcal{D}) = \text{CKh}(\mathcal{D})$.
- $\partial = \partial_{\text{CKh}} + \partial_2 + \partial_3 + \cdots$.

**Theorem (Szabó)**

The homology $\text{Sz}(L) = H(\text{CSz}(\mathcal{D}), \partial)$ is an invariant of $L$. 
The map $\partial_i$ is a sum of maps along $i$-dimensional faces of the cube of resolutions.
Szabó assigns maps to these pictures.

**Conjecture (Seed, Szabó)**

\[ Sz(L) \cong \hat{HF}(\Sigma(-L)) \otimes \hat{HF}(S^1 \times S^2). \]
The invariant: round 2

\( \mathbf{t} = (t_1, t_2, t_3) \) a bridge trisection diagram \( \Sigma \) with each \( t \) is a braid half-plat closure.

**Theorem (S.)**

Let \( \mathcal{A}(\mathbf{t}) = \bigoplus_{i,j} CSz(t_i \bar{t}_j) \).

- \( \mathcal{A}(\mathbf{t}) \) is an \( A_\infty \)-algebra.
- The \( A_\infty \)-chain homotopy type of this algebra is an invariant of the trisection of the surface.
- The binary multiplication \( m_2 \) is still Khovanov’s map, so \( H(\mathcal{A}(\mathbf{t})) \) is still determined by \( g(\Sigma) \).

In principle, computable by computer – no holomorphic disks, PDEs.
The construction uses hyperboxes of chain complexes due to Manolescu and Ozsváth (also Baldwin and Seed).
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Higher configurations represent homotopies between compositions of lower compositions.

Goal: arrange all of these homotopies into one structure.
The construction: a hyperbox

Khovanov, Szabó

$x$-axis: handles in column two

$y$-axis: handles in column one
Apply compression:
THE PROOF

Construct maps of (systems of) hyperboxes for tri-plane moves. Show that they induce maps of $A_\infty$-algebras. Uses functoriality of Szabó’s homology theory. (S. ’17)
The simplest example

If \( t = (t_1, t_2, t_3) \) is the \( b = 1 \) trisection of \( S^2 \), then \( A(t) \) has no higher multiplications.

The next simplest example

If \( t' \) is a stabilization of \( t \), then \( A(t) \) has higher multiplications.

Stabilization will always increase rank.
Let $t'$ be a stabilization of $t$.

😍 Construct maps $A(t) \rightarrow A(t')$. 
Let $t'$ be a stabilization of $t$.

 качальн Construct maps $A(t) \to A(t')$. 
Let $t'$ be a stabilization of $t$. Then:

- Construct maps $\mathcal{A}(t) \to \mathcal{A}(t')$.
- $H(\mathcal{A}(t))$:
  - has associative algebra structure determined by $b, c$
  - has $A_\infty$-structure equivalent to $\mathcal{A}(t)$. 

Isotopy class invariants?
Let $t'$ be a stabilization of $t$.

.construct maps $A(t) \to A(t')$.

• has associative algebra structure determined by $b, c$
• has $A_\infty$-structure equivalent to $A(t)$.

Compare to Heegaard Floer constructions.
Come to Bill Olsen’s talk!
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**Conjecture (Seed, Szabó)**

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Replace \( CSz \) with \( \hat{HF} \)

Trisection diagram \( \approx \) Heegaard multidiagrams. Goal: invariant from counting holomorphic polygons.

Bridge trisection diagram \( \rightarrow \) trisection diagram for branched double cover of \( S^4 \) along \( \Sigma - \mathcal{A}(t) \) contains the four-manifold invariant?
Connections to Heegaard Floer invariants

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Thanks!